

# Study of radial vibrations in cylindrical bone in the framework of transversely isotropic poroelasticity

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## Abstract

In this paper, a cylindrical thick-walled hollow bone filled with marrow is modeled as a transversely isotropic poroelastic thick-walled hollow cylinder filled with a fluid, and analytical solutions for radial vibrations in the same have been obtained. Considering the boundaries to be stress free, frequency equations are obtained in the cases of bone without marrow and bone with marrow for permeable and impermeable boundaries. Limiting the case when the ratio of thickness to inner radius is very small is investigated numerically. The accessible information of bony material has been used for the numerical evaluation. Attenuation versus ratio of outer and inner radii of bone at various anisotropic ratios are computed. Phase velocity and group velocity are computed as a function of wavenumber. These values are depicted graphically and then discussed.

## Keywords

Attenuation, bone, frequency equation, group velocity, poroelasticity, phase velocity, radial vibrations, ratio of radii, wave propagation

## 1. Introduction

The study of wave propagation in poroelastic solids, in general, is of practical importance in the fields of engineering, medicine, and geophysics, particularly, for investigations in destructive research areas. Poroelasticity has been widely used to model biological tissues, such as bone, cartilage, arterial walls, and brain as almost all tissues have some kind of fluid in their pores. Musculoskeletal complications, namely osteoporosis (OP) not only affects bones mineral density (BMD) but also quality. The major consequence of OP is a reduction in bone density leading to bone fracture. The strength of bone is an important parameter for bone quality. Wave velocity and attenuation through a bone depends on the poroelastic parameters of bone, which are directly related to the strength of the bone. Bone tends to exhibit linearly elastic behavior over small strains, if not, it exhibits anisotropy (Humphrey and DeLange, 2007). Biot's equations (Biot 1955, 1956) are employed to investigate the bone and the equations are derived from the consolidation theory to investigate wave propagation in the bones under the assumption that bone exhibits transversely isotropic behavior

(Nowinski and Davis, 1971). In Nowinski and Davis (1972), bones are modeled as anisotropic poroelastic bodies. Cowin (1999) described fluid flow in bone tissues employing the poroelasticity theory. Analytic solutions for electromechanical wave propagation in a cylindrical poroelastic long bone with cavity are studied (Ahmed and Abd-Alla, 2002). Fluid flowing through anisotropic, poroelastic bone models in the opposite direction is investigated (Steck et al., 2003). Dispersion of axially symmetric waves in cylindrical bone filled with marrow is investigated under the assumption that bone exhibits isotropic poroelastic behavior (Malla Reddy et al., 2011). In Malla Reddy et al. (2011), a comparative dispersive study is made between the bone without marrow and the bone with marrow. Gilbert et al. (2012) made a quantitative

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ultrasound model of the bone with blood as the interstitial pore fluid where they considered the Biot model treats the medium as an elastic frame with interstitial pore fluid. From the classical poroelasticity perspective, the following are a few papers on circular cylinders. Approximate formulas for wave lengths associated with the free vibrations of a thick-walled hollow elastic cylinder are examined by McFadden (1954). The analytical solutions for pore pressure and stress fields for an inclined bore hole and the cylinder are investigated (Cui et al., 1997; Abousleiman and Cui, 1998). Axially and non-axially symmetric vibrations of thick-walled hollow poroelastic cylinders and wave propagation in an empty cylindrical bore are investigated (Malla Reddy and Tajuddin, 1999, 2000) in the frame work of Biot's theory of wave propagation in fluid saturated porous media. Solorza and Sahay (2004) developed expressions for torsional resonance and temporal attenuation frequencies for fully saturated poroelastic circular cylinders. Poromechanic analysis of a fully saturated transversely isotropic hollow cylinder is made (Kanj et al, 2003; Kanj and Abousleiman, 2004). Ahmed Shah (2008) studied axially symmetric vibrations of fluid-filled poroelastic circular cylindrical shells. In Ahmed Shah (2008), the frequency equations of axially symmetric vibrations propagating in a fluid-filled and an empty isotropic poroelastic bore, are each investigated for pervious and impervious surfaces. Radial vibrations of thick-walled hollow isotropic poroelastic cylinders are studied (Tajuddin and Shah, 2010). Flexural wave propagation in coated poroelastic cylinders with reference to fretting fatigue is studied (Ahmed Shah, 2011). In Tajuddin and Shah (2010), the dissipative coefficient is neglected due to the mathematical complexity. To the best of the authors' knowledge, radial vibrations in thick-walled transversely isotropic cylindrical bone in the presence of dissipation are not yet investigated. In the present paper, bone with marrow is modeled as a transversely isotropic poroelastic cylinder filled with fluid. The pertinent governing equations in the case of radial vibrations are derived. The frequency equations in the presence of dissipation are derived in the cases of bone without marrow and bone with marrow. In a limiting case, the attenuation as a function of the ratio of radii is computed for various anisotropic ratios.

The rest of the paper is organized as follows. In Section 2, cylindrical bone without marrow is considered. In Section 3, first behavior of waves in marrow independently from the solid bone is investigated, next the bone with marrow is examined. The limiting cases and numerical results are discussed in Section 4. Finally, conclusions are given in Section 5.

## 2. Cylindrical bone without marrow

Consider cylindrical coordinate system  $(r, \theta, z)$  with  $z$ -axis along the axis of poroelastic cylindrical bone. Let the bone be homogeneous and transversely isotropic. Assume that  $z$ -axis is along the axis of rotational symmetry. Hence, in this case eight constants are involved.

The constitutive stress-strain relations for a transversely isotropic poroelastic solid (Abousleiman and Cui, 1998) are given below. The notations are adopted from the said reference.

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{r\theta} \\ \tau_{\theta z} \\ \tau_{rz} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 & 0 & 0 \\ M_{12} & M_{11} & M_{13} & 0 & 0 & 0 \\ M_{13} & M_{13} & M_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{55} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ \gamma_{r\theta} \\ \gamma_{\theta z} \\ \gamma_{rz} \end{bmatrix} - \begin{bmatrix} \alpha \\ \alpha \\ \alpha' \\ 0 \\ 0 \\ 0 \end{bmatrix} p, \quad (1)$$

$$p = M[\varepsilon - \alpha(e_{rr} + e_{\theta\theta}) - \alpha'e_{zz}]$$

In equation (1),  $e_{ij}$  and  $\gamma_{jk}$  are strain components;  $\sigma_{ij}$  and  $\tau_{jk}$  are stress components in the cylindrical coordinate system, respectively (Abousleiman and Cui, 1998);  $p$  is the pore pressure,  $\alpha$  and  $\alpha'$  are Biot's effective stress coefficients in the isotropic plane ( $r$ - $\theta$  plane) and in the  $z$ -direction, respectively;  $M$  is Biot's modulus,  $\varepsilon$  is the variation of fluid content per unit reference volume, and  $M_{jk}$  are components of the drained elastic modulus which depend on  $E$ ,  $E'$ ,  $\nu$ ,  $\nu'$ ,  $G$  and  $G'$ .  $E$  and  $\nu$  are drained Young's modulus and Poisson's ratio in the isotropic plane,  $E'$  and  $\nu'$  are similar quantities as that of  $E$  and  $\nu$  pertaining to the direction of the axis of symmetry,  $G$  and  $G'$  are the shear modulus related to the direction of the isotropic plane and axis of symmetry, respectively. The symbols  $\alpha$  and  $\alpha'$  can be expressed in terms of  $M_{jk}$  and the bulk modulus of solid constituents  $K_s$  and computed as in Abousleiman and Cui (1998). Young's modulus ( $E$ ) and Poisson's ratio ( $\nu$ ) (which leads to all necessary  $M_{jk}$ ),  $M$  and  $K_s$  are suffice to compute all the above parameters. For given anisotropic ratios of  $N_E = \frac{E'}{E}$  and  $N_\nu = \frac{\nu'}{\nu}$  (Kanj and Abousleiman, 2004),  $E'$  and  $\nu'$  can be determined. Different  $N_E$  and  $N_\nu$  ratios define different degrees of anisotropy.

The equations of motion in transversely isotropic poroelastic solids in the presence of dissipation ( $b$ ) are

not readily available in the literature, hence the same are derived from the constitutive relations (Abousleiman and Cui, 1998) and equations of equilibrium, and are given below:

$$\begin{aligned}
 & \left( \frac{M_{11} - M_{12}}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) + \frac{M_{44}}{2r^2} \frac{\partial^2}{\partial \theta^2} + \frac{M_{55}}{2} \frac{\partial^2}{\partial z^2} \right. \\
 & + \left( M_{12} + M\alpha^2 + \frac{M_{11} - M_{12}}{2} \right) \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) u_r \\
 & + \left( \frac{M_{44}}{2r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) \right) u_\theta + \frac{M_{55}}{2} \frac{\partial^2 u_z}{\partial r \partial z} - \alpha M \frac{\partial \varepsilon}{\partial r} \\
 & = \frac{\partial^2}{\partial t^2} (\rho_{11} u_r + \rho_{12} U_r) + b \frac{\partial}{\partial t} (u_r - U_r), \\
 & \left( \left( M_{12} + M\alpha^2 + \frac{M_{44}}{2r} \right) \frac{\partial^2}{\partial r \partial \theta} \right. \\
 & + \left( (M_{11} + M\alpha^2) \frac{1}{r} + \frac{M_{44}}{r^2} \right) \frac{\partial}{\partial \theta} \Big) u_r \\
 & + \left( M_{44} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \right. \\
 & + (M_{11} + M\alpha^2) \frac{1}{r} \frac{\partial^2}{\partial \theta^2} + \frac{M_{55}}{2} \frac{\partial^2}{\partial z^2} \Big) u_\theta \\
 & + \left( \left( M_{13} + M\alpha\alpha' + \frac{M_{55}}{2r} \right) \frac{\partial^2}{\partial \theta \partial z} \right) u_z - \alpha M \frac{\partial \varepsilon}{\partial \theta} \\
 & = \frac{\partial^2}{\partial t^2} (\rho_{11} u_\theta + \rho_{12} U_\theta) + b \frac{\partial}{\partial t} (u_\theta - U_\theta), \\
 & \left( \left( M_{13} + M\alpha\alpha' + \frac{M_{55}}{2} \right) \left( \frac{\partial^2}{\partial \theta \partial z} + \frac{1}{r} \frac{\partial}{\partial z} \right) \right) u_r \\
 & + \left( \left( M_{13} + M\alpha\alpha' + \frac{M_{55}}{2} \right) \frac{1}{r} \frac{\partial^2}{\partial r \partial z} \right) u_\theta \\
 & + \left( \frac{M_{55}}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + (M_{13} + M\alpha^2) \frac{\partial^2}{\partial z^2} \right) u_z \\
 & - \alpha' M \frac{\partial \varepsilon}{\partial \theta} = \frac{\partial^2}{\partial t^2} (\rho_{11} u_z + \rho_{12} U_z) + b \frac{\partial}{\partial t} (u_z - U_z), \\
 & M \frac{\partial \varepsilon}{\partial r} - \alpha M \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) u_r \\
 & - \frac{\alpha M}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_\theta - M\alpha' \frac{\partial^2 u_z}{\partial r \partial z} \\
 & = \frac{\partial^2}{\partial t^2} (\rho_{12} u_r + \rho_{22} U_r) - b \frac{\partial}{\partial t} (u_r - U_r), \\
 & \frac{M}{r} \frac{\partial \varepsilon}{\partial \theta} - \frac{\alpha M}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) u_r - \frac{\alpha M}{r} \frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{M\alpha'}{r} \frac{\partial^2 u_z}{\partial \theta \partial z} \\
 & = \frac{\partial^2}{\partial t^2} (\rho_{12} u_\theta + \rho_{22} U_\theta) - b \frac{\partial}{\partial t} (u_\theta - U_\theta), \\
 & M \frac{\partial \varepsilon}{\partial z} - \alpha M \frac{\partial}{\partial z} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) u_r - \frac{\alpha M}{r} \frac{\partial^2 u_\theta}{\partial \theta \partial z} - M\alpha' \frac{\partial^2 u_z}{\partial z^2} \\
 & = \frac{\partial^2}{\partial t^2} (\rho_{12} u_z + \rho_{22} U_z) - b \frac{\partial}{\partial t} (u_z - U_z),
 \end{aligned} \tag{2}$$

where  $\vec{u}(u_r, u_\theta, u_z)$  and  $\vec{U}(U_r, U_\theta, U_z)$  are solid and fluid displacements,  $e$  and  $\varepsilon$  are the dilatations of solid and fluid, respectively;  $\rho_{jk}$  are mass coefficients.

Consider the transversely isotropic thick-walled cylindrical bone with inner radius  $r_1$ , and outer radius  $r_2$ . Non-zero displacement components of solid and fluid for radial vibrations are  $\vec{u}(u_r, 0, 0)$  and  $\vec{U}(U_r, 0, 0)$ , respectively. These displacements are functions of  $r$  and time  $t$  only. Equations (2) in cylindrical polar coordinates when the solid and fluid displacement components  $u_r$  and  $U_r$  are independent of  $\theta$  and  $z$  reduce to

$$\begin{aligned}
 & \frac{M_{11} - M_{12}}{2} \left( \nabla^2 - \frac{1}{r^2} \right) u_r + \left( M_{12} + M\alpha^2 + \frac{M_{11} - M_{12}}{2} \right) \frac{\partial e}{\partial r} - \alpha M \frac{\partial \varepsilon}{\partial r} \\
 & = \frac{\partial^2}{\partial t^2} (\rho_{11} u_r + \rho_{12} U_r) + b \frac{\partial}{\partial t} (u_r - U_r), \\
 & M \frac{\partial \varepsilon}{\partial r} - \alpha M \frac{\partial e}{\partial r} = \frac{\partial^2}{\partial t^2} (\rho_{12} u_r + \rho_{22} U_r) - b \frac{\partial}{\partial t} (u_r - U_r) \tag{3}
 \end{aligned}$$

In equation (3),  $\nabla^2$  is the Laplacian. The solid and fluid displacement components can be evaluated from equation (3) representing plane harmonic vibrations. In this case, the displacements of solid  $u_r$  and fluid  $U_r$  in the radial direction are taken as follows:

$$\begin{aligned}
 u_r(r, t) &= f(r) e^{i\omega t}, \\
 U_r(r, t) &= F(r) e^{i\omega t}
 \end{aligned} \tag{4}$$

Substituting equation (4) in equation (3), then equation (3) becomes

$$\begin{aligned}
 (M_{11} + M\alpha^2) \Delta f - \alpha M \Delta F &= -\omega^2 (K_{11} f + K_{12} F), \\
 -\alpha M \Delta f + M \Delta F &= -\omega^2 (K_{12} f + K_{22} F)
 \end{aligned} \tag{5}$$

Solving the above equations, there is obtained

$$F = X_1^2 \Delta f - X_2^2 f \tag{6}$$

where

$$\begin{aligned}
 X_1^2 &= \frac{-MM_{11}}{\omega^2(MM_{12} - \alpha M_{22})}, \quad X_2^2 = \frac{MK_{11} + \alpha MK_{12}}{MK_{12} + \alpha MM_{22}}, \\
 \Delta &= \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2}
 \end{aligned} \tag{7}$$

Substituting  $F$  into equation (5), there is obtained

$$\Delta^2 f + X_3^2 f_1 + X_4^2 f = 0 \tag{8}$$

where

$$X_3^2 = \omega^2 \left( \frac{(M_{11} + M\alpha^2)M_{22} + MM_{11} - \alpha MK_{12}}{MM_{11}} \right),$$

$$X_4^2 = \omega^2 \left( \frac{K_{11}K_{22} - K_{12}^2}{MM_{11}} \right).$$

Equation (8) can also be written as

$$\Delta^2 f + (\xi_1^2 + \xi_2^2)\Delta f + (\xi_1^2 \xi_2^2)f = 0 \quad (9)$$

where  $\xi_1^2$  and  $\xi_2^2$  are given by  $\xi_1^2 + \xi_2^2 = X_3^2$ ,  $\xi_1^2 \xi_2^2 = X_4^2$ . Equation (9) can also be written as

$$(\Delta + \xi_1^2)f = 0, \quad (\Delta + \xi_2^2)f = 0 \quad (10)$$

Solving equation (10), there is obtained  $f$  for  $\xi_1$  and  $\xi_2$ . Similarly,  $F$  can be obtained. Substituting these in equation (4), the displacement components are as follows.

$$u_r(r, t) = (c_1 J_1(\xi_1 r) + c_2 Y_1(\xi_1 r) + c_3 J_1(\xi_2 r) + c_4 Y_1(\xi_2 r))e^{i\omega t},$$

$$U_r(r, t) = -(c_1 \delta_1^2 J_1(\xi_1 r) + c_2 \delta_1^2 Y_1(\xi_1 r) + c_3 \delta_2^2 J_1(\xi_2 r) + c_4 \delta_2^2 Y_1(\xi_2 r))e^{i\omega t} \quad (11)$$

where  $c_1, c_2, c_3$  and  $c_4$  are arbitrary constants,  $\omega$  is the frequency of wave,  $J_n$  and  $Y_n$  are Bessel functions of first and second kind of order  $n$ , respectively, and  $\delta_j, \xi_j$  ( $j = 1, 2$ ) are

$$\delta_j^2 = \left( (MK_{11} + \alpha MK_{12}) - \frac{\xi_j^2}{\omega^2} MM_{11} \right) \times (MK_{12} + \alpha MK_{22})^{-1}, \quad (j = 1, 2),$$

$$\xi_1 = \frac{(L_1 + L_2)^{1/2}}{L_3}, \quad \xi_2 = \frac{(L_1 - L_2)^{1/2}}{L_3},$$

$$L_1 = \omega^2((M_{11} + M\alpha^2)K_{22} + MK_{11} + 2\alpha MK_{12}),$$

$$L_2 = \left( \omega^4((M_{11} + M\alpha^2)K_{22} + MK_{11} + 2\alpha MK_{12})^2 - 4\omega^2 MM_{11}(K_{11}K_{22} - K_{12}^2) \right)^{1/2},$$

$$L_3 = (2MM_{11})^{1/2} \quad (12)$$

In equation (12),  $K_{11} = \rho_{11} - \frac{i\eta}{\omega}$ ,  $K_{12} = \rho_{12} + \frac{i\eta}{\omega}$  and  $K_{22} = \rho_{22} - \frac{i\eta}{\omega}$ .

Using these displacement components into stress-displacement relations (Biot, 1956), the relevant stress  $\sigma_{rr}$  and fluid pressure  $p$  are obtained, which are

$$\sigma_{rr} = (c_1 A_{11}(r) + c_2 A_{12}(r) + c_3 A_{13}(r) + c_4 A_{14}(r))e^{i\omega t},$$

$$p = (c_1 A_{21}(r) + c_2 A_{22}(r) + c_3 A_{23}(r) + c_4 A_{24}(r))e^{i\omega t} \quad (13)$$

where

$$A_{11}(r) = \frac{-2}{r}(M\delta_1^2(1 - \alpha) - \frac{1}{2}(M_{11} + M_{12} + 2M\alpha(\alpha - 1))J_1(\xi_1 r) + (M\delta_1^2(\alpha - 1) - (M_{11} + M\alpha(\alpha - 1)))\xi_1 J_1(\xi_1 r),$$

$$A_{21}(r) = \frac{-2}{r}(M\delta_1^2 + \alpha M)J_1(\xi_1 r) + (M\delta_1^2 + \alpha M)\xi_1 J_1(\xi_1 r),$$

$A_{12}(r), A_{22}(r)$  are similar expressions as  $A_{11}(r), A_{21}(r)$  with  $J_1, J_2$  replaced by  $Y_1, Y_2$ , respectively,

$A_{13}(r), A_{23}(r)$  are similar expressions as  $A_{11}(r), A_{21}(r)$  with  $\xi_1, \delta_1$  replaced by  $\xi_2, \delta_2$ , respectively,

$A_{14}(r), A_{24}(r)$  are similar expressions as  $A_{11}(r), A_{21}(r)$  with  $J_1, J_2, \xi_1, \delta_1$  replaced by  $Y_1, Y_2, \xi_2, \delta_2$ , respectively

$$(14)$$

The stress-free boundary conditions in the case of a permeable boundary of thick-walled hollow (without marrow) bone are

$$\sigma_{rr} + p = 0, \quad p = 0 \quad \text{at } r = r_1 \text{ and } r = r_2 \quad (15)$$

In the case of impermeable boundary, there follows

$$\sigma_{rr} + p = 0, \quad \frac{\partial p}{\partial r} = 0 \quad \text{at } r = r_1 \text{ and } r = r_2 \quad (16)$$

Equations (13) and (15) together give a system of four homogeneous equations in four constants  $c_1, c_2, c_3$  and  $c_4$ . In order to obtain a non-trivial solution of this system, the coefficients matrix must be singular. Accordingly, the frequency equation for permeable boundary is

$$\begin{vmatrix} A_{11}(r_1) & A_{12}(r_1) & A_{13}(r_1) & A_{14}(r_1) \\ A_{21}(r_1) & A_{22}(r_1) & A_{23}(r_1) & A_{24}(r_1) \\ A_{11}(r_2) & A_{12}(r_2) & A_{13}(r_2) & A_{14}(r_2) \\ A_{21}(r_2) & A_{22}(r_2) & A_{23}(r_2) & A_{24}(r_2) \end{vmatrix} = 0 \quad (17)$$

Similarly, the frequency equation for impermeable boundary is

$$\begin{vmatrix} A_{11}(r_1) & A_{12}(r_1) & A_{13}(r_1) & A_{14}(r_1) \\ B_{21}(r_1) & B_{22}(r_1) & B_{23}(r_1) & B_{24}(r_1) \\ A_{11}(r_2) & A_{12}(r_2) & A_{13}(r_2) & A_{14}(r_2) \\ B_{21}(r_2) & B_{22}(r_2) & B_{23}(r_2) & B_{24}(r_2) \end{vmatrix} = 0 \quad (18)$$

where,

$$B_{21}(r) = \frac{2}{r}(M\delta_1^2 + \alpha M)\xi_1 J_2(\xi_1 r),$$

$B_{22}(r)$  are similar expressions as  $B_{21}(r)$  with  $J_2$  replaced by  $Y_2$ , respectively,

$B_{23}(r)$  are similar expressions as  $B_{21}(r)$  with  $\xi_1, \delta_1$  replaced by  $\xi_2, \delta_2$ , respectively,

$B_{24}(r)$  are similar expressions as  $B_{21}(r)$  with  $J_2, \xi_1, \delta_1$  replaced by  $Y_2, \xi_2, \delta_2$ , respectively.

If the transverse isotropy effect is neglected, that is, poroelastic cylindrical bone is isotropic, the frequency equations (17) and (18) coincide with that of the paper (Tajuddin and Shah, 2010).

### 3. Cylindrical bone with marrow

Bone marrow is a key component of the circulatory system, producing the blood cells that support the body's immune system. It is a kind of fluid resembling blood, which behaves like a non-Newtonian thinning fluid. For the radial vibrations in the marrow, the displacement potential function  $\phi$  satisfies the wave equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{1}{c_f^2} \frac{\partial^2 \phi}{\partial t^2} \quad (19)$$

where  $c_f$  is the velocity of the sound in the marrow. The solution  $\phi$  is given by  $\phi = J_0(r\omega/c_f)e^{i\omega t}$ . The pressure  $p_f (= \rho_f \frac{\partial^2 \phi}{\partial t^2})$  and radial displacement  $D_r (= \frac{\partial \phi}{\partial r})$  are computed. The ratio of the pressure to radial displacement is

$$\frac{p_f}{D_r} = -\frac{\rho_f \omega^2 J_0(r\omega/c_f)}{\omega/c_f J_1(r\omega/c_f)} \quad (20)$$

The boundary conditions to be satisfied on the surfaces  $r = r_1$  and  $r = r_2$  to be stress-free in a bone filled with marrow for a permeable boundary are

$$\begin{aligned} \frac{\sigma_{rr} + p}{u} &= -\frac{p_f}{D_r}, \quad p = 0, \text{ at } r = r_1 \text{ and } \sigma_{rr} + p = 0, \\ p &= 0 \text{ at } r = r_2 \end{aligned} \quad (21)$$

The first boundary condition presents a condition at the fluid-solid interface, while the other conditions remain the same as in the earlier section. The boundary conditions for the impermeable boundary are

$$\begin{aligned} \frac{\sigma_{rr} + p}{u} &= -\frac{p_f}{D_r}, \quad \frac{\partial p}{\partial r} = 0, \text{ at } r = r_1 \text{ and } \sigma_{rr} + p = 0, \\ \frac{\partial p}{\partial r} &= 0 \text{ at } r = r_2 \end{aligned} \quad (22)$$

For a non-trivial solution, the matrix of coefficients must be singular. Accordingly, the frequency equations in the case of a permeable boundary is

$$\begin{vmatrix} A'_{11}(r_1) & A'_{12}(r_1) & A'_{13}(r_1) & A'_{14}(r_1) \\ A'_{21}(r_1) & A'_{22}(r_1) & A'_{23}(r_1) & A'_{24}(r_1) \\ A_{11}(r_2) & A_{12}(r_2) & A_{13}(r_2) & A_{14}(r_2) \\ A_{21}(r_2) & A_{22}(r_2) & A_{23}(r_2) & A_{24}(r_2) \end{vmatrix} = 0 \quad (23)$$

where

$$\begin{aligned} A'_{11}(r) &= -A_{11}(r)\omega/c_f J_1(r\omega/c_f) + \rho_f \omega^2 J_0(r\omega/c_f) J_1(\xi_1 r), \\ A'_{12}(r) &= -A_{12}(r)\omega/c_f J_1(r\omega/c_f) + \rho_f \omega^2 J_0(r\omega/c_f) Y_1(\xi_1 r), \\ A'_{13}(r) &= -A_{13}(r)\omega/c_f J_1(r\omega/c_f) + \rho_f \omega^2 J_0(r\omega/c_f) J_1(\xi_2 r), \\ A'_{14}(r) &= -A_{14}(r)\omega/c_f J_1(r\omega/c_f) + \rho_f \omega^2 J_0(r\omega/c_f) Y_1(\xi_2 r), \\ A'_{2j}(r) &= A_{2j}(r), \quad j = 1, 2, 3, 4 \end{aligned} \quad (24)$$

The frequency equation in the case of an impermeable boundary is

$$\begin{vmatrix} A'_{11}(r_1) & A'_{12}(r_1) & A'_{13}(r_1) & A'_{14}(r_1) \\ B_{21}(r_1) & B_{22}(r_1) & B_{23}(r_1) & B_{24}(r_1) \\ A_{11}(r_2) & A_{12}(r_2) & A_{13}(r_2) & A_{14}(r_2) \\ B_{21}(r_2) & B_{22}(r_2) & B_{23}(r_2) & B_{24}(r_2) \end{vmatrix} = 0 \quad (25)$$

The frequency equations (17), (18), (23) and equation (25) will be examined numerically in a limiting case.

### 4. Numerical results

In the presence of dissipation ( $b$ ), each element of frequency equations (17), (18), (23) and equation (25) consist of Bessel's functions and the argument of Bessel functions are complex valued, hence giving mathematical complexity in numerical evaluation. To reduce complexity, the limiting case is considered. The limiting case when  $\frac{h}{r_1} \ll 1$ , (i.e.  $\xi_1 r_1, \xi_1 r_2, \xi_2 r_1, \xi_2 r_2 \gg 1$ ) is considered so that asymptotic approximations for Bessel's function (Abramowitz and Stegun, 1965) can be used. Under this condition, cylindrical bone reduces to thin cylindrical poroelastic shell type bone. The frequency equation in the case of bone without marrow for a permeable boundary is

$$\begin{vmatrix} C_{11}(r_1) & C_{12}(r_1) & C_{13}(r_1) & C_{14}(r_1) \\ C_{21}(r_1) & C_{22}(r_1) & C_{23}(r_1) & C_{24}(r_1) \\ C_{11}(r_2) & C_{12}(r_2) & C_{13}(r_2) & C_{14}(r_2) \\ C_{21}(r_2) & C_{22}(r_2) & C_{23}(r_2) & C_{24}(r_2) \end{vmatrix} = 0 \quad (26)$$



where  $C_{jl}(r) = c_{jl}(r) + ic'_{jl}(r)$ ,  $j = 1, 2$ ;  $l = 1, 2, 3, 4$ ,  $i = \sqrt{-1}$ . The expressions for  $c_{jl}(r)$ ,  $c'_{jl}(r)$ ,  $j = 1, 2$ ,  $l = 1, 2, 3, 4$ , are expressed in Appendix A. Similarly, the frequency equation in the case of bone without marrow for an impermeable boundary is

$$\begin{vmatrix} C_{11}(r_1) & C_{12}(r_1) & C_{13}(r_1) & C_{14}(r_1) \\ D_{21}(r_1) & D_{22}(r_1) & D_{23}(r_1) & D_{24}(r_1) \\ C_{11}(r_2) & C_{12}(r_2) & C_{13}(r_2) & C_{14}(r_2) \\ D_{21}(r_2) & D_{22}(r_2) & D_{23}(r_2) & D_{24}(r_2) \end{vmatrix} = 0 \quad (27)$$

where  $D_{2l}(r) = d_{2l}(r) + id'_{2l}(r)$ ,  $l = 1, 2, 3, 4$ ,  $i = \sqrt{-1}$ . The expressions for  $d_{2l}(r)$ ,  $d'_{2l}(r)$ ,  $l = 1, 2, 3, 4$ , are expressed in Appendix A. Under the condition,  $\frac{h}{r_1} \ll 1$ , cylindrical bone with marrow reduces to thin cylindrical poroelastic shell type bone with marrow. Skull, nasal, hip bone, shoulder blades and ribs are of shell type. These bones are thin and generally curved and have red bone marrow rather than both red and yellow bone marrow. The frequency equation (23) in the case of bone with marrow for permeable boundary after substituting the asymptotic approximations would be

$$\begin{vmatrix} P_{11}(r_1) & P_{12}(r_1) & P_{13}(r_1) & P_{14}(r_1) \\ P_{21}(r_1) & P_{22}(r_1) & P_{23}(r_1) & P_{24}(r_1) \\ C_{11}(r_2) & C_{12}(r_2) & C_{13}(r_2) & C_{14}(r_2) \\ C_{21}(r_2) & C_{22}(r_2) & C_{23}(r_2) & C_{24}(r_2) \end{vmatrix} = 0 \quad (28)$$

where  $P_{1l}(r) = p_{1l}(r) + ip'_{1l}(r)$ ,  $P_{2l}(r) = C_{2l}(r)$ ,  $l = 1, 2, 3, 4$ ,  $i = \sqrt{-1}$ . The expressions for  $p_{1l}(r)$ ,  $p'_{1l}(r)$ ,  $l = 1, 2, 3, 4$  are given in Appendix B. Similarly, the frequency equation for an impermeable boundary is

$$\begin{vmatrix} P_{11}(r_1) & P_{12}(r_1) & P_{13}(r_1) & P_{14}(r_1) \\ D_{21}(r_1) & D_{22}(r_1) & D_{23}(r_1) & D_{24}(r_1) \\ C_{11}(r_2) & C_{12}(r_2) & C_{13}(r_2) & C_{14}(r_2) \\ D_{21}(r_2) & D_{22}(r_2) & D_{23}(r_2) & D_{24}(r_2) \end{vmatrix} = 0 \quad (29)$$

where the expressions  $P_{1l}(r)$ ,  $D_{2l}(r)$ ,  $C_{1l}(r)$ ,  $l = 1, 2, 3, 4$  are given in Appendix A and B. The frequency equations (26)–(29) are investigated numerically. The material constants  $M_{11}$ ,  $M_{12}$ ,  $M_{13}$  involves  $E$ ,  $E'$ ,  $\nu$  and  $\nu'$ . The values of Young's modulus ( $E$ ), Poisson's ratio ( $\nu$ ), Skempton pore pressure coefficient ( $B$ ), undrained Poisson's ratio ( $\nu_u$ ) and dissipative coefficient are respectively taken to be 14.58 GPa, 0.32, 0.4, 0.33, and  $0.17 \times 10^{-6}$  GPa/m<sup>2</sup>, respectively, as suggested in Cowin (1999). In Steck et al. (2003), it is assumed that the anisotropic ratios of bone vary in the neighbourhood of one. The bulk modulus of solid constituents ( $K_s$ ) and the Biot's modulus ( $M$ ) can be determined

from the following relations (Kanj and Aboalsleiman, 2004):

$$K_s = \frac{B(1 + \nu_u)E}{3B(1 - 2\nu)(1 + \nu_u) - 9(\nu_u - \nu)},$$

$$M = \frac{B^2(1 - 2\nu)(1 + \nu_u)^2 E}{9(1 - 2\nu_u)(1 + \nu)(\nu_u - \nu)}.$$

The values of mass coefficients  $\rho_{jk}$  are computed as in Nowinski and Davis (1972).

The velocity of sound in marrow ( $c_f$ ) and density of marrow ( $\rho_f$ ) are taken to be 93 m/s and 900 kg/m<sup>3</sup>, respectively (Malla Reddy et al., 2011). Employing these values in the frequency equations, an implicit relation between complex frequency ( $\omega$ ) and the ratio ( $g = \frac{r_2}{r_1}$ ) is obtained. The attenuation coefficient ( $Q^{-1}$ ) (Solorza and Sahay, 2004) is given by

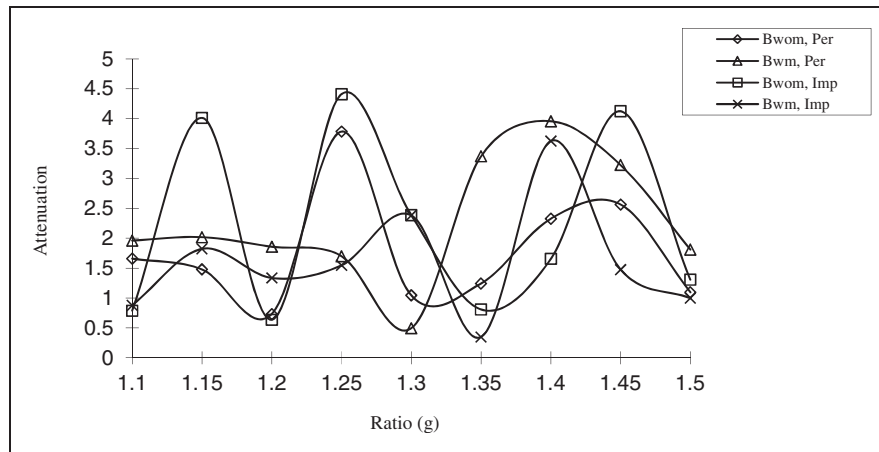
$$Q^{-1} = \frac{2\text{Im}(\omega)}{\text{Re}(\omega)} \quad (30)$$

The phase velocity ( $c_p$ ) and group velocity ( $c_g$ ) are given by

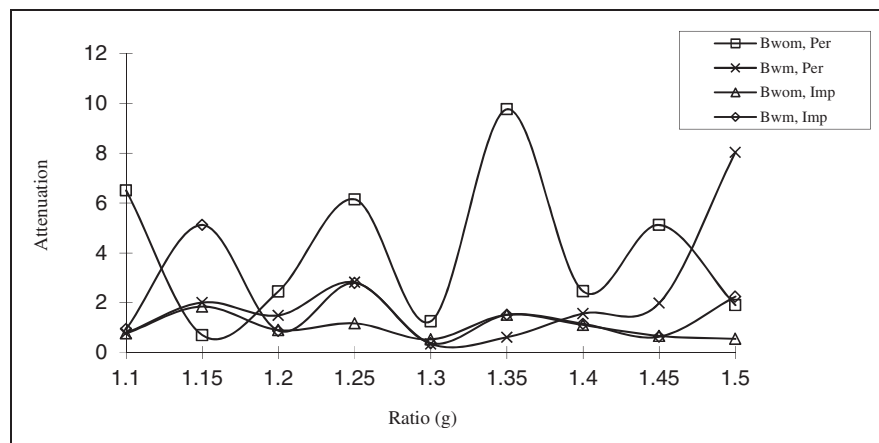
$$c_p = \text{Real part } (\omega/k) \quad (31)$$

$$c_g = \frac{d\omega}{dk} \quad (32)$$

Attenuation against the ratio ( $g$ ) is computed in the cases of bone without marrow and bone filled with marrow for permeable and impermeable boundaries. Phase velocity and group velocity are computed as a function of wavenumber in the case of bone without marrow for a permeable boundary. The numerical values are depicted in Figures 1–6. The notations Bwm, Bwom, Per and Imp used in the figures to represent the bone with marrow case, bone without marrow case, permeable boundary case, and impermeable boundary case, respectively. Figure 1 depicts the variation of attenuation against the ratio ( $g$ ) when the anisotropic ratios  $N_E = 1$  and  $N_\nu = 0.5$  in the cases of bone without marrow and bone with marrow for permeable and impermeable boundaries. From this figure, it is clear that the attenuation values in the case of bone with marrow are, in general, greater than that of bone without marrow in the case of a permeable boundary. Also, it is seen that the attenuation values in the case of bone with marrow are, in general, less than that of bone without marrow in the case of an impermeable boundary. That is, the trend is reversed for an impermeable boundary. Figure 2 shows the attenuation values against the ratio ( $g$ ) when the anisotropic ratios  $N_\nu = 1$  and  $N_E = 0.5$  in the cases of bone without marrow and bone with marrow for permeable and



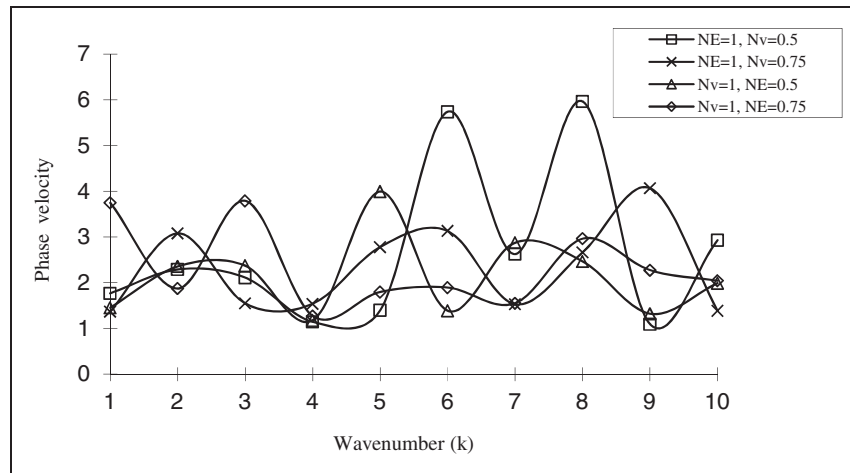
**Figure 1.** Variation of attenuation against the ratio ( $g$ ) when the anisotropic ratios  $N_E = 1$  and  $N_v = 0.5$  in the cases of bone without marrow and bone with marrow for permeable and impermeable boundaries.



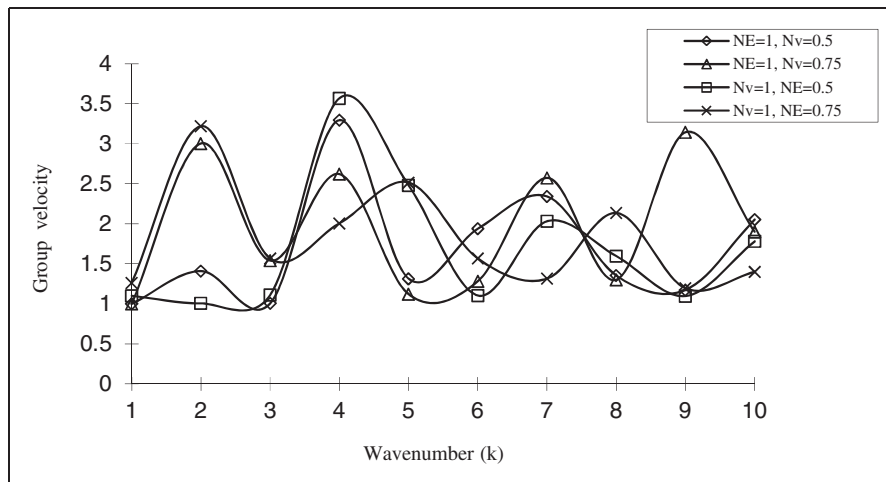
**Figure 2.** Variation of attenuation against the ratio ( $g$ ) when the anisotropic ratios  $N_v = 1$  and  $N_E = 0.5$  in the cases of bone without marrow and bone with marrow for permeable and impermeable boundaries.

impermeable boundaries. From Figure 2, it is observed that attenuation values in the case of bone with marrow are, in general, less than that of bone without marrow in the case of a permeable boundary. It is also seen that attenuation values in the case of bone with marrow are, in general, greater than that of bone without marrow in the case of an impermeable boundary. From Figures 1 and 2, it is clear that the nature of surface does have an influence over attenuation values in the cases of bone without marrow and bone with marrow. Figures 3 and 4 depict the variation of phase velocity and group velocity with wavenumber ( $k$ ) in the case of bone without marrow for a permeable boundary when the ratio  $g = 1.1$ . From Figures 3 and 4, it is observed that phase velocity and group velocity values at anisotropic ratio  $N_v = 0.5$  are, in general, greater than that of values at  $N_v = 0.75$  when  $N_E = 1$  and values at  $N_E = 0.5$  are, in general, less than that of values at

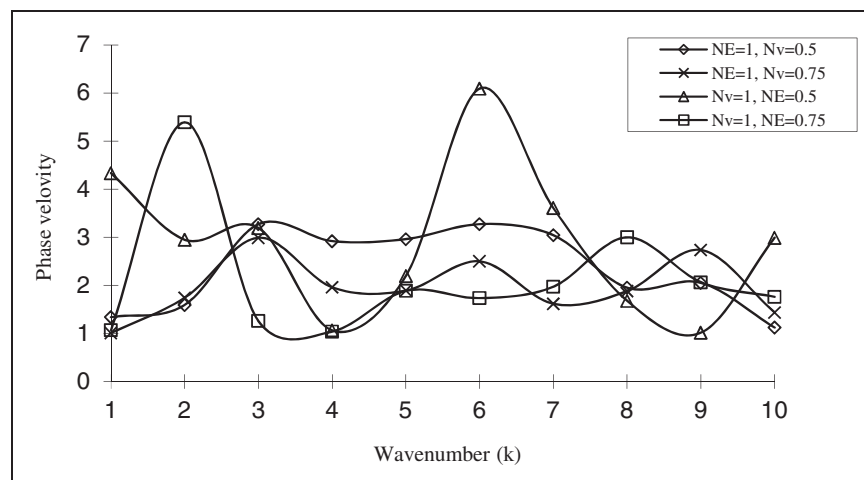
$N_E = 0.75$  when  $N_v = 1$ . Figures 5 and 6 depict the variation of phase velocity and group velocity with wavenumber ( $k$ ) in the case of bone with marrow for a permeable boundary when the ratio  $g = 1.1$ . From Figures 5 and 6, it is seen that phase velocity values at anisotropic ratio  $N_v = 0.5$  are, in general, greater than that of values at  $N_v = 0.75$  when  $N_E = 1$  but group velocity values at anisotropic ratio  $N_v = 0.5$  are, in general, less than that of values at  $N_v = 0.75$  when  $N_E = 1$ . Similar conclusions are made in the case of anisotropic ratio  $N_E = 0.5, 0.75$  when  $N_v = 1$ . From Figures 3–6, it is clear that, for given anisotropic ratios, the trend of phase velocity and group velocity is the same in the bone without marrow case but different in the bone with marrow case. From all the figures, it is found that dispersive behaviors in bone without marrow and bone with marrow cases are different in phenomena.



**Figure 3.** Variation of phase velocity with wavenumber ( $k$ ) in the case of bone without marrow for permeable boundary when the ratio  $g = 1.1$ .

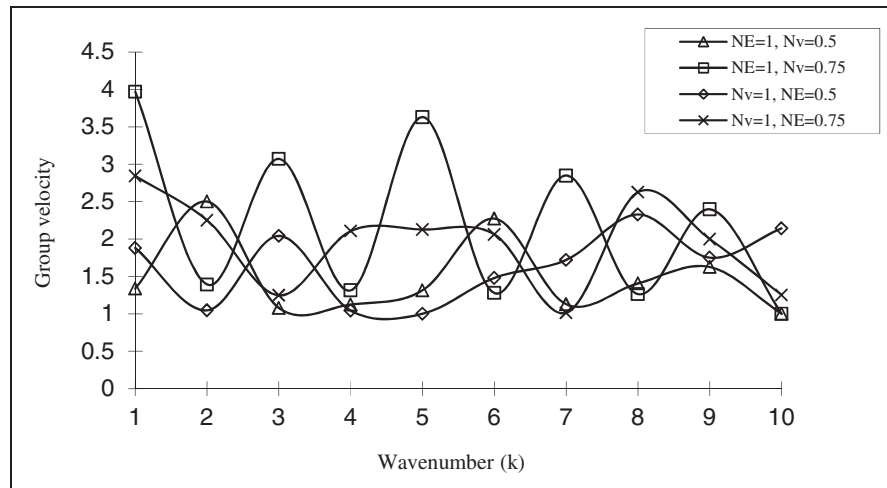


**Figure 4.** Variation of group velocity with wavenumber ( $k$ ) in the case of bone without marrow for permeable boundary when the ratio  $g = 1.1$ .



**Figure 5.** Variation of phase velocity with wavenumber ( $k$ ) in the case of bone with marrow for permeable boundary when the ratio  $g = 1.1$ .





**Figure 6.** Variation of group velocity with wavenumber ( $k$ ) in the case of bone with marrow for permeable boundary when the ratio  $g = 1.1$ .

This discrepancy can be exploited in the medical diagnosis to examine the level of bone. When the ratio of outer and inner radii of bone is fixed, the phase velocity values are, in general, greater than that of group velocity in both the cases of a permeable boundary at various anisotropic ratios. Attenuation, phase velocity, and group velocity values are affected with anisotropic ratios in either case.

## 5. Conclusion

Thick-walled cylindrical bone filled with marrow is modeled as a transversely isotropic thick-walled cylindrical poroelastic cylinder filled with fluid in the framework of Biot's poroelasticity theory. The pertinent equations of motion are derived. A comparative study has been made between the bone without marrow and the bone with marrow for the cases of permeable and impermeable boundaries. The limiting case, when the ratio of thickness to inner radius is very small, is discussed. In this limiting case, thick-walled cylindrical bone reduces to thin shell type bone. The available data of bony materials has been used for numerical evaluation. Attenuation versus ratio of outer and inner radii of bone at various anisotropic ratios are computed. Phase velocity and group velocity are computed as a function of wavenumber at various anisotropic ratios in the cases of bone without marrow and bone with marrow for a permeable boundary at various anisotropic ratios. From the figures, it is observed that the nature of the surface does have an influence over attenuation values in the cases of bone without marrow and bone with marrow. From the figures, it is also clear that, for given anisotropic ratios, the trend of phase velocity and group velocity is the same in the bone without marrow case but different in the bone with

marrow case. From all the figures, it is found that dispersive behaviors in bone without marrow and bone with marrow cases are different in phenomena. This discrepancy can be exploited in the medical diagnosis to examine the level of bone. When the ratio of outer and inner radii of bone is fixed, the phase velocity values are, in general, greater than that of group velocity in both cases of permeable boundary at various anisotropic ratios. Attenuation, phase velocity, and group velocity values are affected with anisotropic ratios in either case.

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## Appendix A

$$c_{11}(r) = \frac{-2}{r}(M(1-\alpha)a_1 - \frac{1}{2}(M_{11} + 2M\alpha(\alpha-1) + M_{12})D_1)b_1 - ((M_{11} + M\alpha(\alpha-1))a_2 + M(1-\alpha)a_3)b_2,$$

$$c_{12}(r) = \frac{2}{r}(M(1-\alpha)a_4 - \frac{1}{2}(M_{11} + 2M\alpha(\alpha-1) + M_{12})D_2)b_1 - ((M_{11} + M\alpha(\alpha-1))a_2 - M(1-\alpha)a_6)b_2,$$

$$c_{21}(r) = \frac{-2}{r}(Ma_1 - \alpha MD_1)b_1 + (\alpha Ma_2 + Ma_3)b_2,$$

$$c_{22}(r) = \frac{-2}{r}(Ma_4 + \alpha MD_2)b_1 + (\alpha Ma_5 + Ma_6)b_2,$$

$c_{13}(r)$ ,  $c_{14}(r)$  are similar expressions as  $c_{11}(r)$ ,  $c_{12}(r)$  with  $a_1, a_2, a_3, b_1, b_2, D_1, D_2$  replaced by  $a_7, a_8, a_9, b_3, b_4, D_3, D_4$ , respectively,

$$a_1 = \delta_{11}(T_{11} \cos(\pi/8) + T_{12} \sin(\pi/8)) + \delta_{12}(T_{12} \cos(\pi/8) - T_{11} \sin(\pi/8)),$$

$$a_2 = L_1(\cos T_1(T_{13} \cos(\pi/8) + T_{14} \sin(\pi/8)) - \sin T_1(T_{14} \cos(\pi/8) - T_{13} \sin(\pi/8))),$$

$$a_3 = (T_{13} \cos(\pi/8) + T_{14} \sin(\pi/8))(\delta_{11}l_1 \cos T_1 + \delta_{12}l_1 \sin T_1) - (T_{13} \cos(\pi/8) + T_{14} \sin(\pi/8)) \times (\delta_{11}l_1 \cos T_1 + \delta_{12}l_1 \sin T_1),$$

$$a_4 = \delta_{11}(T_{21} \cos(\pi/8) + T_{22} \sin(\pi/8)) + \delta_{12}(T_{24} \cos(\pi/8) - T_{23} \sin(\pi/8)),$$

$$a_5 = l_1(\cos T_1(T_{23} \cos(\pi/8) + T_{24} \sin(\pi/8)) - \sin T_1(T_{24} \cos(\pi/8) - T_{23} \sin(\pi/8))),$$

$$a_6 = (T_{23} \cos(\pi/8) + T_{24} \sin(\pi/8)) \times (\delta_{11}l_1 \cos T_1 + \delta_{12}l_1 \sin T_1) - (T_{24} \cos(\pi/8) - T_{23} \sin(\pi/8))(\delta_{11}l_1 \sin T_1 - \delta_{12}l_1 \cos T_1),$$

$$b_1 = \sqrt{\frac{2}{l_1 \pi r}}(\cos^2(\pi/8) - \sin^2(\pi/8))^{-1} \sin(3\pi/4),$$

$$b_2 = \sqrt{\frac{2}{l_1 \pi r}}(\cos^2(\pi/8) - \sin^2(\pi/8))^{-1},$$

$$D_1 = T_{11}(\cos(\pi/8) + T_{12} \sin(\pi/8)),$$

$$D_2 = T_{21}(\cos(\pi/8) + T_{22} \sin(\pi/8)),$$

$$l_1 = (d_1^2 + d_2^2)^{\frac{1}{2}}, \quad T_1 = \frac{1}{2} \tan^{-1} \left( \frac{d_2}{d_1} \right),$$

$$d_1 = \frac{\omega^2(M_{11} + M\alpha^2)\rho_{22} + M\rho_{11} + 2\alpha M\rho_{12}}{2((M_{11} + M\alpha^2)M - \alpha^2 M^2)} + l_2 \cos T_2,$$

$$d_2 = \frac{-\omega b((M_{11} + M\alpha^2) + M(1 + \alpha))}{2((M_{11} + M\alpha^2)M - \alpha^2 M^2)} + l_2 \sin T_2,$$

$$l_2 = (d_3^2 + d_4^2)^{\frac{1}{4}}, \quad T_2 = \frac{1}{2} \tan^{-1} \left( \frac{d_4}{d_3} \right),$$

$$d_3 = \omega^4((M_{11} + M\alpha^2)\rho_{22} + M\rho_{11} + 2\alpha M\rho_{12})^2 \\ + \frac{b^2}{\omega^2}(M_{11} + M\alpha^2 + M - 2\alpha M)^2 \\ - 4\omega^2(\rho_{11}\rho_{22} - \rho_{12}^2),$$

$$d_4 = 4\omega b(\rho_{11} + 2\rho_{12} + \rho_{22})M_{11} - \frac{2b}{\omega}((M_{11} + M\alpha^2)\rho_{22} \\ + M\rho_{11} + 2\alpha M\rho_{12})(M_{11} + M\alpha^2 + M - 2\alpha M),$$

$$\delta_{11} = \frac{M\rho_{11} + \alpha M\rho_{12} - MM_{11}l_1^2 \cos(2T_1)/\omega^2}{M\rho_{12} + \alpha M\rho_{22}},$$

$$\delta_{12} = \frac{MM_{11}l_1^2 \sin(2T_1)/\omega^2}{M\rho_{12} + \alpha M\rho_{22}},$$

$$T_{11} = \sin(l_1 \cos T_1 r) \cosh(l_1 \sin T_1 r) \\ - \cos(l_1 \cos T_1 r) \sinh(l_1 \sin T_1 r),$$

$$T_{12} = \cos(l_1 \cos T_1 r) \sinh(l_1 \sin T_1 r) \\ + \sin(l_1 \cos T_1 r) \cosh(l_1 \sin T_1 r),$$

$$T_{13} = \sin(l_1 \cos T_1 r) \cosh(l_1 \sin T_1 r) \\ + \cos(l_1 \cos T_1 r) \sinh(l_1 \sin T_1 r),$$

$$T_{14} = \cos(l_1 \cos T_1 r) \sinh(l_1 \sin T_1 r) \\ - \sin(l_1 \cos T_1 r) \cosh(l_1 \sin T_1 r),$$

$$T_{21} = x_1 + x_3 + \frac{15}{8r}((y_1 x_1 - y_2 x_2) - (x_3 y_1 + y_2 x_4)),$$

$$T_{22} = x_4 - x_2 - \frac{15}{8r}((y_1 x_4 - y_2 x_3) + (x_2 y_1 + y_2 x_1)),$$

$$T_{23} = x_3 - x_1 + \frac{15}{8r}((y_1 x_3 + y_2 x_4) - (x_1 y_1 + y_2 x_2)),$$

$$T_{24} = x_4 + x_2 + \frac{15}{8r}((y_1 x_4 - y_2 x_3) - (x_1 y_2 + y_2 x_1)),$$

$$x_1 = \cos(l_1 \cos T_1 r) \cosh(l_1 \sin T_1 r),$$

$$x_2 = \sin(l_1 \cos T_1 r) \sinh(l_1 \sin T_1 r),$$

$$x_3 = \sin(l_1 \cos T_1 r) \cosh(l_1 \sin T_1 r),$$

$$x_4 = \cos(l_1 \cos T_1 r) \sinh(l_1 \sin T_1 r),$$

$$y_1 = \cos T_1, \quad y_2 = \sin T_1,$$

$T_{31}, T_{32}, T_{33}, T_{34}, T_{41}, T_{42}, T_{43}, T_{44}$  are similar expressions as  $T_{11}, T_{12}, T_{13}, T_{14}, T_{21}, T_{22}, T_{23}, T_{24}$  with  $l_1, T_1$  replaced by  $l_3, T_3$ , respectively,  $a_7, a_8, a_9$  are similar expressions as  $a_1, a_2, a_3$  with  $T_{11}, T_{12}, T_{13}, T_{14}, l_1, T_1, b_1, b_2$  replaced by  $T_{31}, T_{32}, T_{33}, T_{34}, l_3, T_3, b_3, b_4$ , respectively,

$b_3, b_4$  are similar expressions as  $b_1, b_2$  with  $l_1$  replaced by  $l_3$ ,

$$D_3 = T_{31} \cos \pi/8 + T_{32} \sin \pi/8,$$

$$D_4 = T_{41} \cos \pi/8 + T_{42} \sin \pi/8,$$

$a_{10}, a_{11}, a_{12}$  are similar expressions as  $a_4, a_5, a_6$  with  $T_{21}, T_{22}, T_{23}, T_{24}, l_1, T_1, b_1, b_2$  replaced by  $T_{41}, T_{42}, T_{43}, T_{44}, l_3, T_3, b_3, b_4$ , respectively,  $\delta_{21}, \delta_{22}$  are similar expressions as  $\delta_{11}, \delta_{12}$  with  $l_1, T_1$  replaced by  $l_3, T_3$ , respectively,

$$a_{13} = \delta_{11}(T_{12} \cos \pi/8 - T_{11} \sin \pi/8)$$

$$- \delta_{12}(T_{11} \cos \pi/8 + T_{12} \sin \pi/8),$$

$$a_{14} = L_2(\sin T_1(T_{13} \cos \pi/8 + T_{14} \sin \pi/8)$$

$$+ \cos T_1(T_{14} \cos \pi/8 - T_{13} \sin \pi/8)),$$

$$a_{15} = (T_{14} \cos \pi/8 - T_{13} \sin \pi/8)(\delta_{11} L_1 \cos T_1$$

$$+ \delta_{12} L_1 \sin T_1) + (T_{13} \cos \pi/8 + T_{14} \sin \pi/8)$$

$$\times (\delta_{11} L_1 \sin T_1 - \delta_{12} L_1 \cos T_1),$$

$$D_5 = T_{12} \cos \pi/8 - T_{11} \sin \pi/8,$$

$$D_7 = T_{32} \cos \pi/8 - T_{31} \sin \pi/8,$$

$c_{13}(r), c_{14}(r)$  are similar expressions as  $c_{11}(r), c_{12}(r)$  with  $a_1, a_2, a_3, b_1, b_2, D_1, D_2, \delta_{11}, \delta_{12}$  replaced by  $a_7, a_8, a_9, b_3, b_4, D_3, D_4, \delta_{21}, \delta_{22}$ , respectively,  $c_{23}(r), c_{24}(r)$  are similar expressions as  $c_{21}(r), c_{22}(r)$  with  $a_4, a_5, a_6, b_1, b_2, D_1, D_2, \delta_{11}, \delta_{12}$  replaced by  $a_{10}, a_{11}, a_{12}, b_3, b_4, D_3, D_4, \delta_{21}, \delta_{22}$ , respectively,

$$l_3 = (d_5^2 + d_6^2)^{\frac{1}{4}}, \quad T_3 = \frac{1}{2} \tan^{-1} \left( \frac{d_6}{d_5} \right),$$

$$d_5 = \frac{\omega^2((M_{11} + M\alpha^2)\rho_{22} + M\rho_{11} + 2\alpha M\rho_{12})}{2M_{11}} - l_2 \cos T_2,$$

$$d_6 = \frac{-\omega b((M_{11} + M\alpha^2) + M + \alpha^2 M^2)}{2M_{11}} - l_2 \sin T_2,$$

$$c'_{11}(r) = \frac{-2}{r}(M(1 - \alpha)a_{13} \\ - \frac{1}{2}(M_{11} + 2M\alpha(\alpha - 1) + M_{12})D_5)b_1 \\ - ((M_{11} + M\alpha(\alpha - 1))a_{14} - M(1 - \alpha)a_{15})b_2,$$

$$c'_{12}(r) = \frac{2}{r}(M(1 - \alpha)a_{16} + \frac{1}{2}(M_{11} + 2M\alpha(\alpha - 1) \\ + M_{12})D_6)b_1 - ((M_{11} + M\alpha(\alpha - 1))a_{17} \\ - M(1 - \alpha)a_{18})b_2,$$

$$c'_{21}(r) = \frac{-2}{r}(Ma_{13} + \alpha MD_5)b_1 + (\alpha Ma_{14} + Ma_{15})b_2,$$

$$c'_{22}(r) = \frac{2}{r}(Ma_{16} + \alpha MD_6)b_1 + (\alpha Ma_{17} + Ma_{18})b_2,$$

$a_{19}, a_{20}, a_{21}$  are similar expressions as  $a_{13}, a_{14}, a_{15}$  with  $T_{11}, T_{12}, T_{13}, T_{14}, l_1, T_1, \delta_{11}, \delta_{12}, D_5$  replaced by  $T_{31}, T_{32}, T_{33}, T_{34}, l_3, T_3, \delta_{21}, \delta_{22}, D_7$ , respectively,  $a_{22}, a_{23}, a_{24}$  are similar expressions as  $a_{16}, a_{17}, a_{18}$  with  $T_{21}, T_{22}, T_{23}, T_{24}, l_1, T_1, \delta_{11}, \delta_{12}, D_6$  replaced by  $T_{41}, T_{42}, T_{43}, T_{44}, l_3, T_3, \delta_{21}, \delta_{22}, D_8$ , respectively,

$$\begin{aligned} a_{17} &= l_1(\cos T_1(T_{23} \cos \pi/8 + T_{24} \sin \pi/8) \\ &\quad - \sin T_1(T_{24} \cos \pi/8 - T_{23} \sin \pi/8)), \\ a_{18} &= (T_{24} \cos \pi/8 - T_{23} \sin \pi/8) \\ &\quad \times (\delta_{11} l_1 \cos T_1 + \delta_{12} l_1 \sin T_1) \\ &\quad + (T_{23} \cos \pi/8 + T_{24} \sin \pi/8) \\ &\quad \times (\delta_{11} l_1 \sin T_1 - \delta_{12} l_1 \cos T_1), \\ D_6 &= T_{14} \cos \pi/8 - T_{13} \sin \pi/8, \\ D_8 &= T_{34} \cos \pi/8 - T_{33} \sin \pi/8, \end{aligned}$$

$c'_{13}(r), c'_{14}(r)$  are similar expressions as  $c'_{11}(r), c'_{12}(r)$  with  $a_{13}, a_{14}, a_{15}, b_1, b_2, D_5, D_6, \delta_{11}, \delta_{12}$  replaced by  $a_{19}, a_{20}, a_{21}, b_3, b_4, D_7, D_8, \delta_{21}, \delta_{22}$ , respectively,  $c'_{23}(r), c'_{24}(r)$  are similar expressions as  $c'_{21}(r), c'_{22}(r)$  with  $a_{16}, a_{17}, a_{18}, b_1, b_2, D_5, D_6, \delta_{11}, \delta_{12}$  replaced by  $a_{22}, a_{23}, a_{24}, b_3, b_4, D_7, D_8, \delta_{21}, \delta_{22}$  respectively,

$$\begin{aligned} d_{21}(r) &= \frac{2}{r}((M\delta_{11} + \alpha M)(D_9 l_2 \cos T_2 - D_{10} l_2 \sin T_2) \\ &\quad + M\delta_{12}(D_{10} l_2 \cos T_2 + D_9 l_2 \sin T_2)), \\ d_{22}(r) &= \frac{2}{r}((M\delta_{11} + \alpha M)(D_{11} l_2 \cos T_2 - D_{12} l_2 \sin T_2) \\ &\quad + M\delta_{12}(D_{11} l_2 \sin T_2 + D_{12} l_2 \cos T_2)), \\ d'_{21}(r) &= \frac{2}{r}((M\delta_{11} + \alpha M)(D_{10} l_2 \cos T_2 + D_9 l_2 \sin T_2) \\ &\quad - M\delta_{12}(D_9 l_2 \cos T_2 - D_{10} l_2 \sin T_2)), \\ d'_{22}(r) &= \frac{2}{r}((M\delta_{11} + \alpha M)(D_{11} l_2 \sin T_2 + D_{12} l_2 \cos T_2) \\ &\quad - M\delta_{12}(D_{11} l_2 \cos T_2 - D_{12} l_2 \sin T_2)), \end{aligned}$$

$$\begin{aligned} D_9 &= T_{21} \cos \pi/8 + T_{22} \sin \pi/8, \\ D_{10} &= T_{22} \cos \pi/8 - T_{21} \sin \pi/8, \\ D_{11} &= T_{23} \cos \pi/8 + T_{24} \sin \pi/8, \\ D_{12} &= T_{24} \cos \pi/8 + T_{23} \sin \pi/8, \\ D_{13} &= T_{41} \cos \pi/8 + T_{42} \sin \pi/8, \\ D_{14} &= T_{42} \cos \pi/8 - T_{41} \sin \pi/8, \\ D_{15} &= T_{43} \cos \pi/8 + T_{44} \sin \pi/8, \\ D_{16} &= T_{44} \cos \pi/8 + T_{43} \sin \pi/8, \end{aligned}$$

$d_{23}(r), d_{24}(r), d'_{23}(r), d'_{24}(r)$  are similar expressions as  $d_{21}(r), d_{22}(r), d'_{21}(r), d'_{22}(r)$  with  $l_2, T_2, D_9, D_{10}, D_{11}, D_{12}, \delta_{11}, \delta_{12}$  replaced by  $l_3, T_3, D_{13}, D_{14}, D_{15}, D_{16}, \delta_{21}, \delta_{22}$ , respectively,

## Appendix B

$$\begin{aligned} p_{11}(r) &= \frac{-\omega}{c_f} J_1\left(\frac{r\omega}{c_f}\right) c_{11}(r) + \rho_f \omega^2 J_0\left(\frac{r\omega}{c_f}\right) b_1 D_1, \\ p_{12}(r) &= \frac{-\omega}{c_f} J_1\left(\frac{r\omega}{c_f}\right) c_{12}(r) + \rho_f \omega^2 J_0\left(\frac{r\omega}{c_f}\right) b_1 D_2, \\ p'_{11}(r) &= \frac{-\omega}{c_f} J_1\left(\frac{r\omega}{c_f}\right) c'_{11}(r) + \rho_f \omega^2 J_0\left(\frac{r\omega}{c_f}\right) b_1 D_5, \\ p'_{12}(r) &= \frac{-\omega}{c_f} J_1\left(\frac{r\omega}{c_f}\right) c'_{12}(r) + \rho_f \omega^2 J_0\left(\frac{r\omega}{c_f}\right) b_1 D_6, \end{aligned}$$

$p_{13}(r), p_{14}(r)$  are similar expressions as  $p_{11}(r), p_{12}(r)$  with  $c_{11}(r), c_{12}(r), D_1, D_2, b_1$  replaced by  $c_{13}(r), c_{14}(r), D_3, D_4, b_3$ , respectively,

$p'_{13}(r), p'_{14}(r)$  are similar expressions as  $p'_{11}(r), p'_{12}(r)$  with  $c'_{11}(r), c'_{12}(r), D_5, D_6, b_1$  replaced by  $c'_{13}(r), c'_{14}(r), D_7, D_8, b_3$  respectively.